Area of a Region

-We will expand the idea of the area under the curve.

-The sum of all n terms $a_1, a_2, a_3, ..., a_n$ is written:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the index of summation

a_i is i-th term of the sum

and the upper and lower bounds of summation are n and 1.

Example

a) $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b) $\sum_{i=0}^{5} (i+1) = 1 + 2 + 3 + 4 + 5 + 6$

c) $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d) $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^1+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

e) $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x$

Properties

a) $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

b) $\sum_{i=1}^{n}\left(a_i + b_i\right) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Summation Formulas

a) $\sum_{i=1}^{n} c = cn$

b) $\sum_{i=1}^{n} i = \dfrac{n\left(n+1\right)}{2}$

c) $\sum_{i=1}^{n} i^2 = \dfrac{n\left(n+1\right)\left(2n+1\right)}{6}$

d) $\sum_{i=1}^{n} i^3 = \dfrac{n^2\left(n+1\right)^2}{4}$

Example

Evaluate $\sum_{i=1}^{n} \dfrac{i+1}{n^2}$ for $n = 10, 100, 1000, 10\ 000$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(i+1\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

$$= \frac{1}{n^2}\left[\frac{n^2+3n}{2}\right]$$

$$= \frac{n+3}{2n}$$

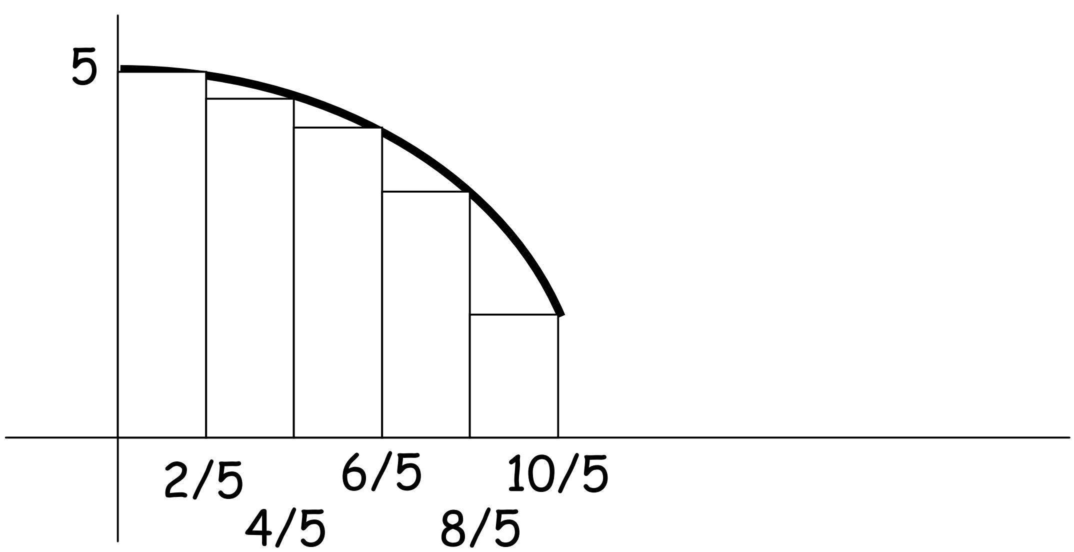
n	$\frac{n+3}{2n}$
10	0.65000
100	0.51500
1,000	0.50150
10,000	0.50015

$$\lim_{n\to\infty}\frac{n+3}{2n} = \frac{1}{2}$$

Area of a Plane Region

Use 5 rectangles to find **two** approximations of the area of the region lying between the graphs of

$f(x) = -x^2 + 5$ and the x-axis between $x = 0, x = 2$



Right endpoints are $\frac{2}{5}i$ where $i = 1, 2, 3, 4, 5$

Width is $\frac{2}{5}$

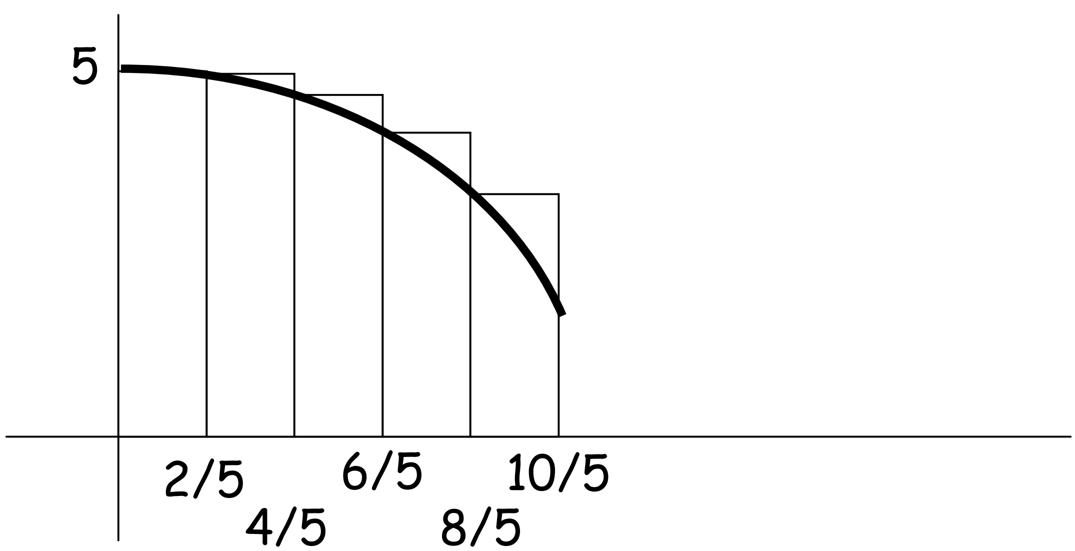
Height is found by evaluating at the endpoints:

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

The sum of the area is:

$$\sum_{i=1}^{5} f\left(\frac{2}{5}i\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \left(\frac{126}{25}\right) = 6.48$$

The area is actually greater!!



Left endpoints are $\frac{2}{5}(i-1)$ where $i = 1, 2, 3, 4, 5$

Width is $\frac{2}{5}$

Height is found by evaluating the left endpoint

$$\sum_{i=1}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \left(\frac{202}{25}\right) = 8.08$$

So,

$$6.48 < A < 8.08$$

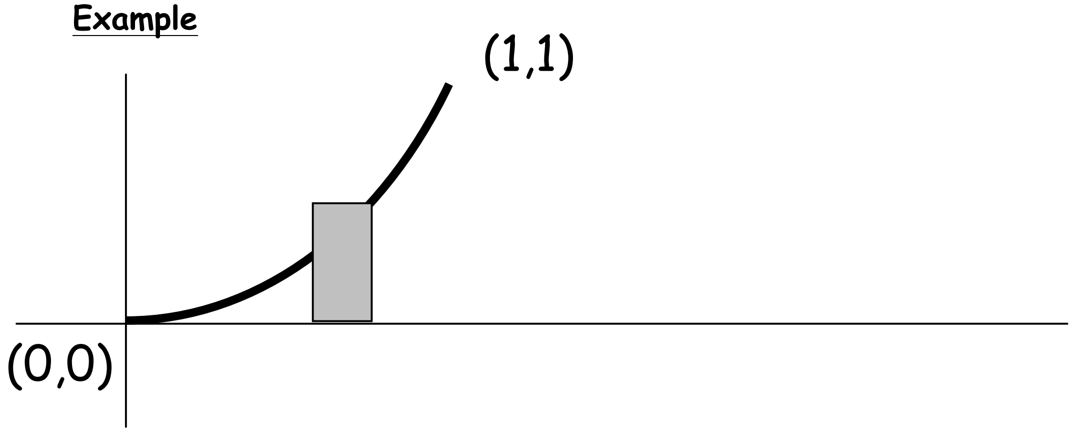
Definition of Area of a Region in the Plane

Let f be continuous and non-negative on the interval $\left[a,b\right]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(c_i\right)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$

Example



Find the area of the region bounded by the graph of $f\left(x\right) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Note that *f* is continuous and non-negative on the interval $\left[0,1\right]$.

Next, partition the interval $\left[0,1\right]$ into *n* subintervals each of width $\Delta x = \frac{1}{n}$.

According to the definition of area, you can choose any x-value in the i-th subinterval. Right endpoints are convenient.

$$c_i = \frac{i}{n}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(c_i\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$$

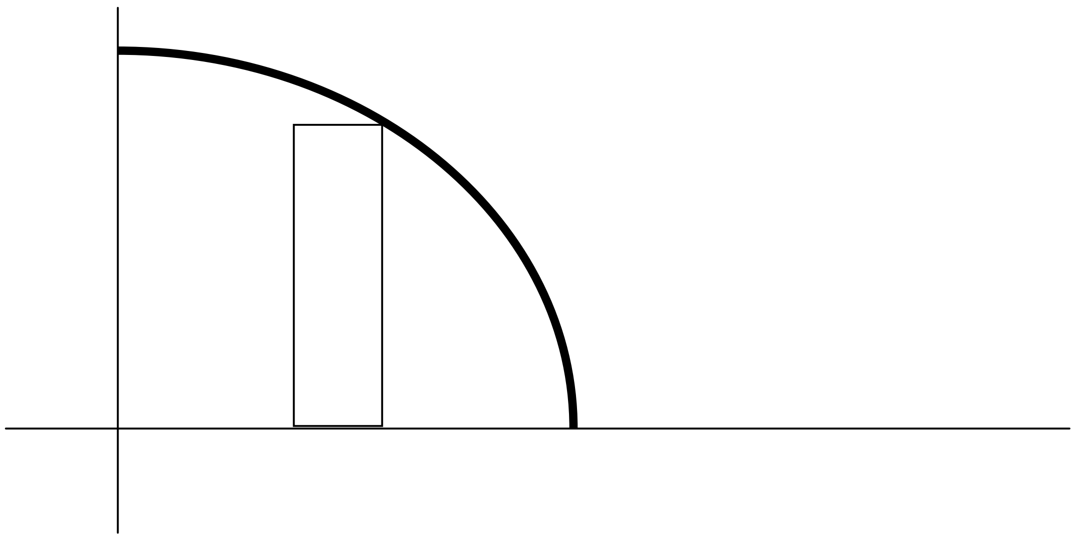
$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2\left(n+1\right)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}$$

Example

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$.



Continuous and non-negative? [Yes!]

$$\Delta x = \frac{1}{n}$$

Right Endpoint:

$$c_i = a + i\Delta x = 1 + \frac{i}{n}$$

Area:

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left[4 - \left(1 + \frac{i}{n} \right)^2 \right] \left(\frac{1}{n} \right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2} \right) \left(\frac{1}{n} \right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}3 - \frac{2}{n^2}\sum_{i=1}^{n}i - \frac{1}{n^3}\sum_{i=1}^{n}i^2\right]$$

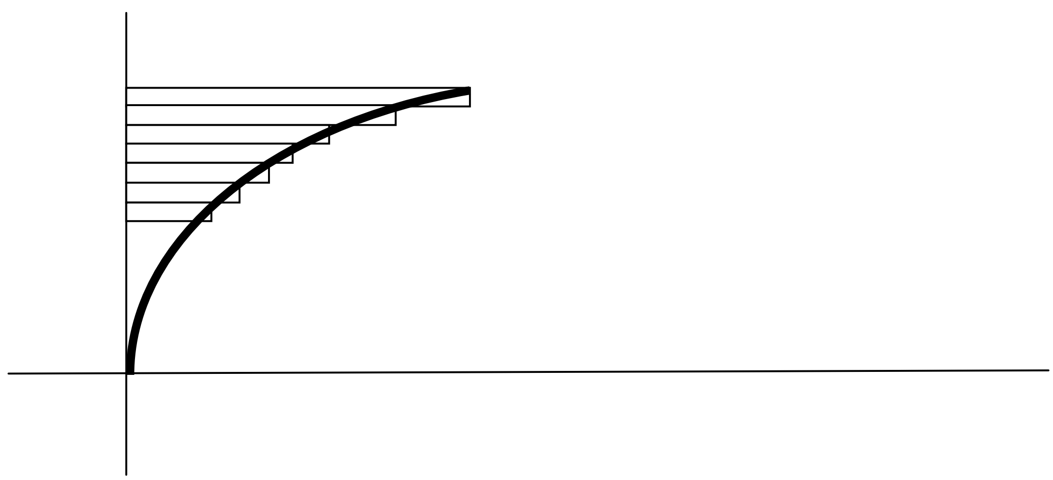
$$= \lim_{n\to\infty}\left[3 - \left(1 - \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right]$$

$$= 3 - 1 - \frac{1}{3}$$

$$= \frac{5}{3}$$

Example

Find the area of the region bounded by the graph of $f\left(y\right) = y^2$ and the y-axis for $0 \le y \le 1$.



Continuous and non-negative [Yes!]

Width $\Delta y = \frac{1}{n}$

$$c_i = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$